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Steady viscous flow near a stationary contact line

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The paper presents the asymptotic solution, near a stationary contact line at a plane boundary, for steady viscous incompressible flow of two immiscible liquids. The eigenvalues which determine this Stokes flow are determined by the contact angle α of the more viscous liquid and the ratio μ of the two viscosities. The dominant eigenvalues are found for all values of α and μ. As $\mu \rightarrow 0$ the results agree with those of Moffatt's (1964) one-phase theory for the case $\mu = 0$ only when $\alpha > 81°$. For $\alpha < 81°$ the two sets of results are qualitatively different. In particular, the eddy structure corresponding to complex eigenvalues occurs only in the α-range (34°, 81°). As μ increases from 0 to 1, this range steadily decreases to zero, which is located at 60°. The transport of energy across the liquid interface is almost always from the obtuse-angled sector to the acute-angled sector, irrespective of α, μ, and the location of the global power supply.

1. Introduction

We consider the steady two-dimensional flow of two immiscible viscous liquids in the immediate neighbourhood of their common stationary contact line with a plane solid boundary. This contact line is taken to be orthogonal to the plane of the flow. We assume that the relevant physical properties of both liquids are constant, homogeneous, and independent of the flow. In particular, we assume that the balance of surface tensions at the contact line implies a well-defined contact angle α, where

$$0 < \alpha < \pi, \tag{1.1}$$

which represents the angle of the sector occupied by the more viscous liquid, and that α takes its hydrostatic value. The exclusion of the end points from the interval (1.1) is a necessary condition for the validity of what follows, since this exclusion makes any bounded curvature of the interface at the contact line irrelevant to the local dynamics of the flow in a sufficiently small domain surrounding the contact line. Thus, the usual relation between the discontinuity in normal stress and the curvature of the interface becomes irrelevant; as does the orientation of the local flow with respect to any external field of force such as gravity. Such matters affect only higher-order approximations to the local flow, which become appreciable only at some distance from the contact line. Asymptotically, therefore, the flow is the same as that in which the interface is accurately plane, and an arbitrary distribution of discontinuity in normal stress is permissible.

With this interpretation, the flow configuration and its associated coordinate system is that shown in figure 1. The appropriate local analysis is thus a

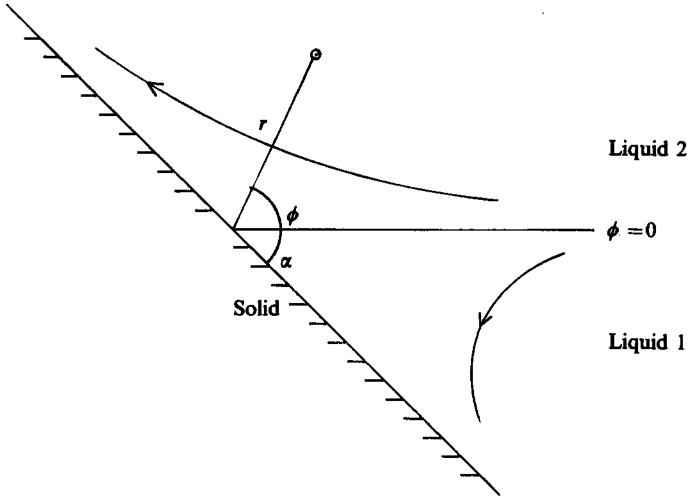

FIGURE 1. Orthogonal section of the flow configuration. The contact line 0 is the axis of a system of cylindrical polar coordinates (r, ϕ) in which the plane $\phi = 0$ represents the interface between the two liquids. The solid boundary is given by $\phi = -\alpha$, $\phi = \pi - \alpha$.

generalization of that developed by Moffatt (1964), and is another example of a two-phase similarity flow (Moffatt & Duffy 1980; Hooper, Duffy & Moffatt 1982). Thus we take the stream function ψ to be everywhere of the form

$$\psi = \mathrm{Re}\{Ar^{\lambda+1}f(\phi)\}, \tag{1.2}$$

where A and λ are (in general complex) constants.

If the rate of dissipation of total energy in any cylinder of unit span enclosing the contact line is to be bounded, which we impose as a condition, then we must have

$$\mathrm{Re}\{\lambda\} > 0, \tag{1.3}$$

which ensures that ψ satisfies the Stokes equation

$$\nabla^4\psi = 0. \tag{1.4}$$

The usual kinematic and dynamic boundary conditions for ψ are

$$f(-\alpha) = f(-0) = f'(-\alpha) = 0; \tag{1.5a}$$

$$f(+0) = f(\pi-\alpha) = f'(\pi-\alpha) = 0; \tag{1.5b}$$

$$f'(-0) - f'(+0) = 0; \tag{1.5c}$$

$$\mu_1 f''(-0) - \mu_2 f''(+0) = 0; \tag{1.5d}$$

where μ_1 and μ_2 $(\mu_1 > \mu_2)$ are the viscosities of the liquids, and the notations $+0$ and -0 represent limits as $\phi \to 0$ from above and below, respectively. In the limit $r \to 0$ no other boundary conditions may be imposed.

Equations (1.4) and (1.5) lead to a standard linear eigenvalue problem of order eight, whose characteristic equation is

$$\mu = \frac{\mu_2}{\mu_1} = -\left\{\frac{\pi-\alpha}{\alpha}\right\} \frac{\{F(2\lambda\alpha) - F(2\alpha)\}\{F^2(\lambda\pi - \lambda\alpha) - F^2(\pi-\alpha)\}}{\{F(2\lambda\pi - 2\lambda\alpha) - F(2\pi - 2\alpha)\}\{F^2(\lambda\alpha) - F^2(\alpha)\}}, \tag{1.6}$$

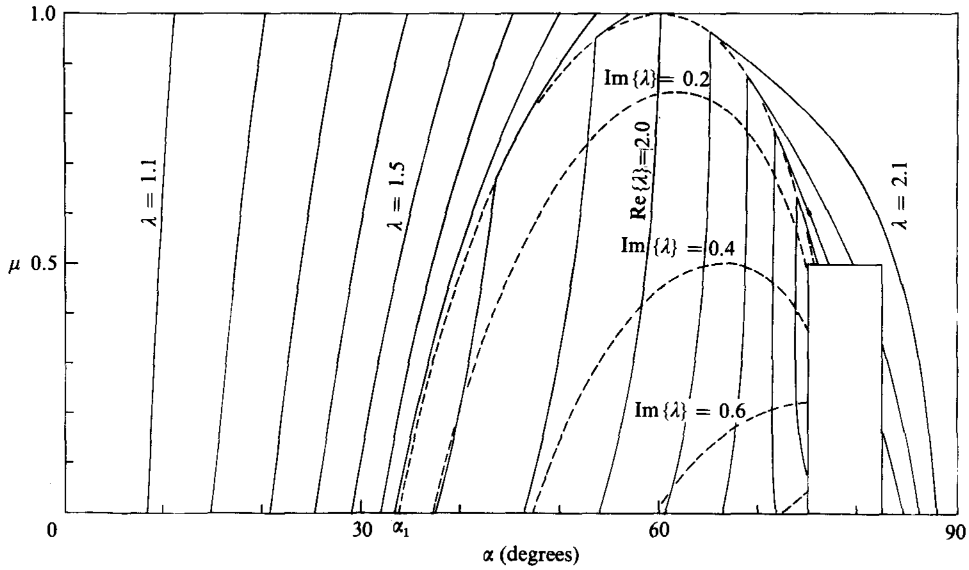

FIGURE 2. Contours of dominant eigenvalues for acute contact angles: ——, Re$\{\lambda\}$, 0.1 intervals; ---, Im$\{\lambda\}$, 0.2 intervals. The envelope of the contours of real λ is the limiting contour Im$\{\lambda\} = 0$.

where
$$F(z) = \frac{\sin}{z}. \tag{1.7}$$

For given real values of α and μ this equation gives the (generally complex) eigenvalues λ. As a check on the algebra, note that the solutions of (1.6) are invariant under the transformation
$$\mu \to \frac{1}{\mu'}, \quad \alpha \to \pi - \alpha', \tag{1.8}$$
as is obviously required by the arbitrariness of the labelling of the two liquids.

2. The dominant eigenvalues

For each given pair of values of α and μ the characteristic equation (1.6) admits an infinite sequence of eigenvalues λ. However, as $r \to 0$ the dominant eigenfunction (1.2) is that whose eigenvalue has the smallest strictly positive real part. Such dominant eigenvalues are easily computed from (1.6) by elementary iterative methods, provided a reasonably good first approximation is available. The purpose of figures 2–4 is to provide such approximations at every relevant point of the (α, μ)-plane, and to indicate the general topography of the function $\lambda(\alpha, \mu)$. Since $\mu_1 > \mu_2$ it is necessary to consider only the rectangular domain
$$0 < \mu \leqslant 1, \quad 0 < \alpha < \pi.$$
In this domain, the figures show the contour map of Re$\{\lambda\}$ and, in the subdomain where λ is complex, the corresponding contour map of Im$\{\lambda\}$.

As figure 2 shows†, the dominant eigenvalues are always real provided $\alpha < \alpha_1 = 33.7°$. Then, for $\alpha_1 < \alpha < 60°$, the eigenvalues remain real only for sufficiently large

† The contours of real λ are taken from Asadullah (1986).

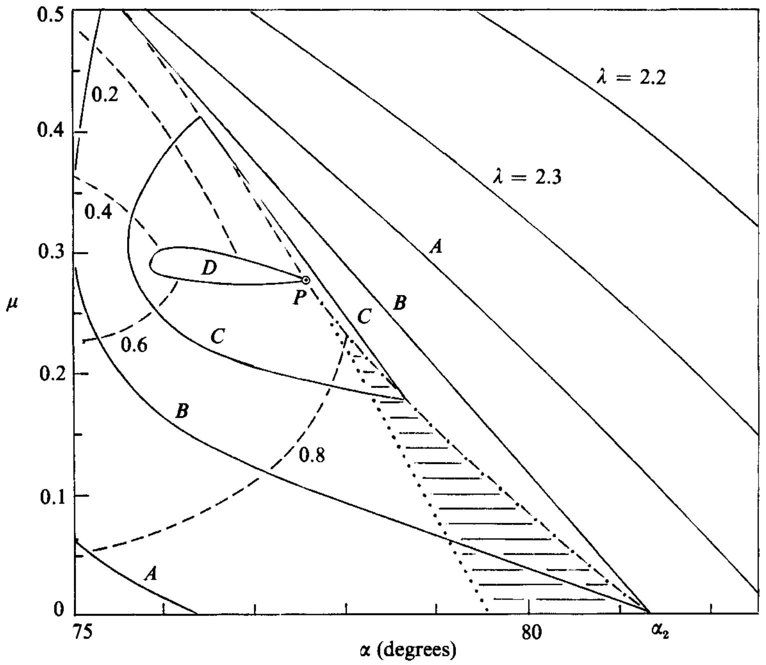

FIGURE 3. Contours of dominant eigenvalues in the sensitive domain of figure 2: ——, Re$\{\lambda\}$, (A) 2.4, (B) 2.511, (C) 2.6, (D) 2.688; - - -, Im$\{\lambda\}$, 0.2 intervals; ......, envelope of real eigenvalues where this is not the boundary of dominant real eigenvalues; -·-, boundary of dominant real eigenvalues, where this is not the envelope of real eigenvalues.

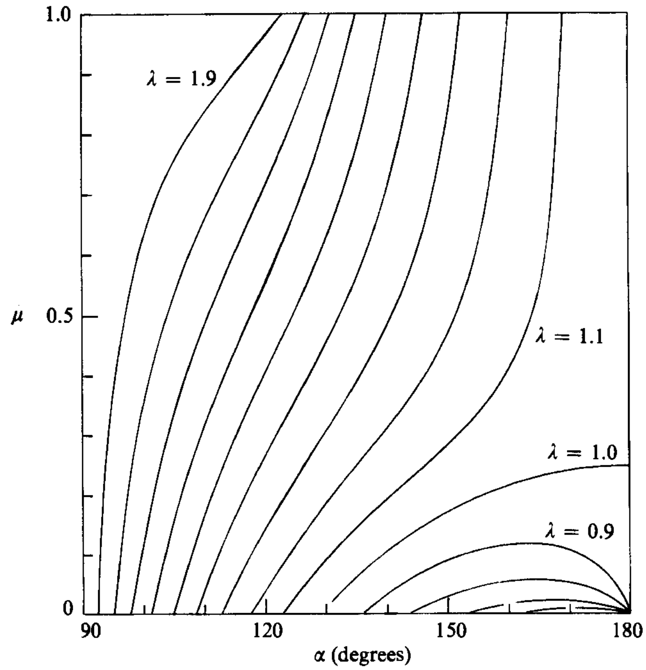

FIGURE 4. Contours of dominant eigenvalues for obtuse contact angles: Re$\{\lambda\}$, 0.1 intervals.

values of μ, the boundary of this 'real' domain being the envelope of the contours of real λ along which the transformation to complex eigenvalues is continuous.

At $\alpha = 60°$ real eigenvalues are impossible apart from the degenerate case, $\mu = 1$, $\lambda = 2$.

For $60° < \alpha < \alpha_2 = 81.3°$ the dominant eigenvalues are again real only for sufficiently large values of μ, but the boundary between the 'real' and 'complex' domains is here considerably more complicated. For $\alpha < 77.5°$, the boundary remains the envelope of the contours of real λ; but at this point ($\alpha = 77.5°$, $\mu = 0.276$, the point P in figure 3), the characteristic equation has two distinct roots with the same real part. One of them is real ($\lambda = 2.688$), and the other has a non-zero imaginary part ($\mathrm{Im}\{\lambda\} = 0.767$). In the shaded area of figure 3, it is the analytic continuation of this second root which has a smaller real part than the real root, and which provides the dominant eigenvalues. Hence there is a new boundary between the domains of real and complex eigenvalues, across which the transition is not only discontinuous, but from the largest value of $\mathrm{Im}\{\lambda\}$ that is possible for any given value of μ. This boundary meets the axis $\mu = 0$ at $\alpha = \alpha_2$; and for values of α greater than this, the dominant eigenvalues are real for all values of μ, as figures 2–4 show.

One further property of the general topography may be noted. The contour $\mathrm{Re}\{\lambda\} = 2.688$, which produces the two distinct roots at the point P, is a closed curve. Within this closed curve, still larger values of $\mathrm{Re}\{\lambda\}$ are dominant until the contours degenerate to a point at $\alpha = 75.9°$, $\mu = 0.293$. At this last point, $\mathrm{Re}\{\lambda\} = 2.716$, which is the largest dominant value in the whole (α, μ)-plane.

3. The case of small viscosity ratio

The earlier work of Dean & Montagnon (1949) and Moffatt (1964) for one-phase flows may be regarded as the limit $\mu = 0$, and their results may be compared with the appropriate limiting form of the present results. Since the analysis requires inertia forces to be negligible in both liquids, it is necessary to regard the limit $\mu \to 0$ in the following form: $\mu_1 \to \infty$ for a fixed value of μ_2. Otherwise the domain of validity of the similarity solution shrinks to zero.

3.1. *The dominant eigenvalues*

When $\mu = 0$ the results are more conveniently represented in the form shown in figure 5. Two entirely separate sequences of roots of the characteristic equation (1.6) are involved. For $\alpha > \alpha_2$ the dominant root is the dominant root of

$$F(2\lambda\alpha) - F(2\alpha) = 0. \tag{3.1}$$

In the one-phase theory presented by Moffatt (1964), (3.1) is the complete characteristic equation. Hence, in this range, the two theories are necessarily in complete agreement.

When $\alpha < \alpha_2$, however, the dominant root of (1.6) is the dominant root of

$$F(\lambda\pi - \lambda\alpha) + F(\pi - \alpha) = 0. \tag{3.2}$$

At the transition point α_2 the roots of (3.2) are complex and do not become real until α has decreased to the branch point α_1. Throughout the range $\alpha < \alpha_2$, therefore, the limit of the two-phase theory as $\mu \to 0$ is different from Moffatt's one-phase theory at $\mu = 0$, and the two theories yield qualitatively different results.

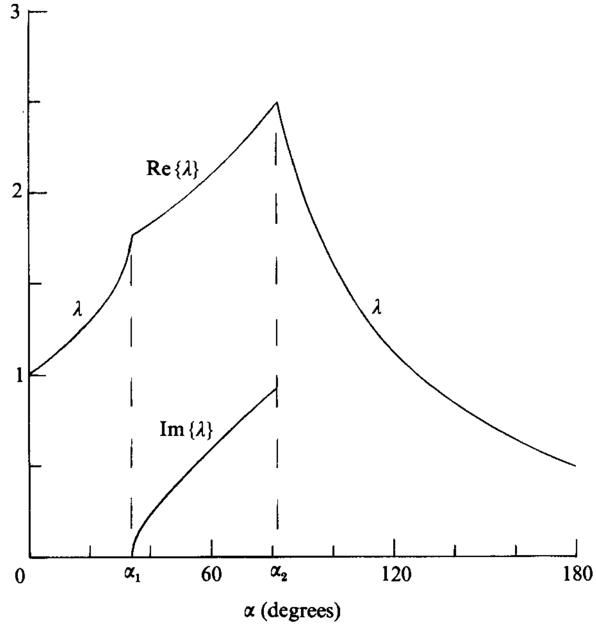

Figure 5. Dominant eigenvalues as $\mu \to 0$.

As Moffatt (1964) has pointed out, the physical manifestation of a complex eigenvalue is an infinite sequence of eddies whose linear size increases linearly with the distance r from the contact line. In the present results, such a structure does not occur for $\alpha < \alpha_1$, and as α increases beyond this range the size of the eddy which contains a fixed value of r steadily decreases from infinity to its value at α_2, at which value the eddy structure suddenly disappears. In Moffatt's results, as α increases from zero, the same eddy size steadily increases from zero to infinity at a particular value of α, at which the solution represents an eddy-free flow. The physical mechanism underlying this difference is discussed in §3.2.

Although the two theories yield different α-ranges for an eddy structure, they both produce rather large values (3900 herein; 5000 in Moffatt) for the minimum ratio of the velocity scales in two successive eddies. Such values raise doubts about whether an eddy zone could ever be observed in a form different from a zone of general stagnation.

3.2. *The two modes of flow*

As $\mu \to 0$ the distinction between the two relevant sequences of eigenvalues is most easily characterized in terms of the mechanical conditions at the interface. When $\alpha > \alpha_2$, so that the dominant eigenvalues are solutions of (3.1), the equations show that $f''(-0)$ vanishes, whereas $f'(-0)$ remains of order unity. They also show that neither $f''(+0)$ nor $f'(+0)$ vanishes; the unbounded ratio $f''(+0)/f''(-0)$ being supported by the vanishing ratio μ. It follows that the velocities in the two liquids are, at any given value of r, of comparable order of magnitude, and that this scale is represented by the velocity at the interface. This mode of flow may therefore reasonably be termed the velocity mode, and is the one discussed by Moffatt (1964).

When $\alpha < \alpha_2$ the effect of the mechanical balance is different. Here, the dominant eigenvalues are solutions of (3.2), and $f''(-0), f'(-0)$ and $f'(+0)$ all vanish, whereas $f''(+0)$ does not, the unbounded ratio again being supported by zero μ. It follows that the more viscous liquid is virtually at rest, and that it is the stresses, not the velocities, that are of comparable magnitude in the two liquids. This mode of flow may therefore reasonably be termed the stress mode. In the stress mode, the flow of the less viscous liquid must be that of flow between two solid boundaries, one of which is the natural solid boundary and the other is the interface.

In confirmation of this interpretation of the stress mode, we note that (3.2) is the characteristic equation obtained by Dean & Montagnon (1949), as well as by Moffatt (1964), for antisymmetric flow of a single fluid between solid boundaries. These same authors also discuss the corresponding case of symmetric flow, for which the characteristic equation is

$$F(\lambda\pi - \lambda\alpha) - F(\pi - \alpha) = 0. \tag{3.3}$$

While it is true that (3.3) is a third limiting form of the full characteristic equation (1.6) as $\mu \to 0$, this form is not normally relevant to the two-phase theory, since the dominant eigenvalue of the sequence is in turn dominated, for every value of α, by the dominant eigenvalue of one of the other two sequences. Clearly, it is the principle of global dominance in the two-phase theory which leads not only to the irrelevance of (3.3), but to the existence of the critical angle α_2 which marks the discontinuous transition between (3.1) and (3.2). Fortuitously the value of α_2 (81°) lies within 2° of another critical angle (79°) obtained by Moffatt (1964)†: namely the branch point at which the solutions of (3.1) change continuously from real to complex values. It is this small difference that leads to the narrowness of the shaded region of figure 3, where it appears that Moffatt's angle is not relevant to the two-phase problem.

3.3. *Transport of energy across the interface*

An interesting aspect of the flow as $\mu \to 0$ concerns the rate of viscous transport of energy across unit area of the interface, E, from the more viscous to the less viscous liquid. For non-zero values of μ, however small, E does not vanish, and its sign becomes a matter of legitimate investigation. When λ is real, it is a simple matter to show that E is negative for $\alpha < \frac{1}{2}\pi$, and positive for $\alpha > \frac{1}{2}\pi$. When λ is complex, which occurs only for acute values of α, the calculation is more awkward since E is an oscillatory function of position along the interface. However, if we adopt the geometrically natural definition of a single eddy as any portion of the flow which is bounded by the streamline $\psi = 0$, so that its interfacial boundary lies between two stagnation points, then the sign of

$$\bar{E} = \int_{\text{one eddy}} E \, \mathrm{d}r$$

again measures, in an average sense, the direction of energy transport. The numerical solutions of the characteristic equation then show that $\bar{E}$ is also always negative, except for a very small subdomain of the (α, μ)-plane which is almost the same as the shaded area of figure 3.

It appears, then, that the transport of energy across the interface is almost always from the obtuse-angled sector to the acute-angled sector, irrespective of the mechanical properties of the two liquids. This result has an interesting implication for any global flow in which the ultimate source of energy resides in a liquid whose

† The value quoted is the corrected value (79.5°) given by Moffatt & Duffy (1980), in agreement with figure 3.

contact angle is acute. Apparently, over much of the interface this liquid must set the other liquid in motion, but, near a contact line, it is the latter which must drive the former liquid.

4. Conclusion

It remains to consider the domain of validity of the local similarity solution (1.2) in a practical flow configuration. The essential constraint is that both the solid boundary and the liquid interface should be approximately plane throughout this domain. If the theory is to be valid right up to the contact line, as ideally it should, then the plane interface must also be the tangent plane at the contact line.

Now it is always possible, by adjusting the inclination of a flat plate which intersects the interface between two liquids in a reservoir, to make this tangent plane horizontal on one side of the plate for any given contact angle of the liquids. If the system were at rest, and the density ratio not very close to unity, gravitational forces would then ensure that the interface would remain almost plane over a substantial domain. For sufficiently small velocities in an induced motion it then seems reasonable to suppose that the departure of the interface from its hydrostatic position can be kept as small as is desired. In such a flow configuration the domain of validity of the local similarity solution at the contact line is enhanced, by orders of magnitude, from that otherwise would be the case. Indeed, the only limitation is the global lengthscale of the reservoir and plate, which must be large compared with the domain of validity. It may even be, as was perhaps implied by Moffatt (1964), that a matching of the inclination of the plate to the contact angle may be unnecessary in some cases, on the grounds that a small region near the contact line, whose linear dimensions are comparable with the small radius of curvature of a meniscus, will not significantly affect the flow in a wedge of much larger extent.

With such an arrangement of the boundaries, it might well be possible to observe critical properties of the flow of two liquids with very different viscosities, which would discriminate between the present results and those of Moffatt (1964). Perhaps the most interesting, and somewhat surprising, difference between the theories lies in our conclusion that Moffatt's results for free-surface flows when $\mu = 0$ cannot represent the limiting flow as $\mu \to 0$ when the contact angle of the more viscous liquid is less than 81°. In particular, Moffatt's corner eddies cannot appear when this contact angle is less than 34°. Some aspects of this marked qualitative difference might be observable in the motion of the less viscous liquid.

Further, the present results show that the qualitative properties of the flow as $\mu \to 0$ are not sensitive to the value of μ. As μ increases from 0 to 0.276 (the point P in figure 3), there remain two distinct modes of flow, and all their qualitative properties remain the same as those for $\mu \to 0$. The only significant changes are in the critical angles (α_1, α_2) at which the main transitions take place. The requirement that the viscosity ratio μ be small may thus be interpreted very loosely.

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